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NUMERICAL SOLUTION OF SIXTH ORDER BOUNDARY VALUE PROBLEMS BY PETROV-GALERKIN METHOD WITH QUINTIC B-SPLINES AS BASIS FUNCTIONS AND SEPTIC B-SPLINES AS WEIGHT FUNCTIONS

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ABSTRACT

In this paper a finite element method involving Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions has been developed to solve a general sixth order boundary value problem with a particular case of boundary conditions. The basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and the Neumann type of boundary conditions are prescribed. The weight functions are also redefined into a new set of weight functions which in number match with the number of redefined basis functions. The proposed method was applied to solve several examples of sixth order linear and nonlinear boundary value problems. The obtained numerical results were found to be in good agreement with the exact solutions available in the literature.

KEYWORDS: Absolute Error, Petrov – Galerk in Method, Quintic B-Spline, Septic B-Spline, Sixth Order Boundary Value Problem

INTRODUCTION

In this paper, we consider a general sixth order linear boundary value problem

$$\begin{aligned} &a_0(x)y^{(6)}(x)+a_1(x)y^{(5)}(x)+a_2(x)y^{(4)}(x)+a_3(x)y'''(x)+a_4(x)y''(x) \\ &+a_5(x)y'(x)+a_6(x)y(x)=b(x), \quad c<x<d \end{aligned} \tag{1}$$

subject to boundary conditions

$$y(c)=A_0,\ y(d)=C_0,\ y'(c)=A_1,\ y'(d)=C_1,\ y''(c)=A_2,\ y''(d)=C_2 \tag{2}$$

where A_0, C_0, A_1, C_1, A_2, C_2 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$, $a_6(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

The sixth order boundary value problems occur in astrophysics [1]. Chandrasekhar [2] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth order. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [3]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on sixth-order boundary value problems by using different methods for numerical solutions. Wazwaz [4] developed the solution of special type of sixth order boundary value problems by using the modified Adomian decomposition method and he provided the solution in the form of a rapidly convergent series. Huan [5] presented variational approach technique to solve a special case of sixth order boundary value problems. Noor et al. [6] presented the variational iteration principle to solve a special case of sixth order boundary value problems after transforming the given differential equation into a system of integral equations. Ghazala and Siddiqi [7] presented the solution of a special case of sixth order boundary value problems by using non-polynomial spline functions. Siddiqi et al. [8], Siddiqi and Ghazala [9] developed quintic spline funtions and septic spline functions techniques to solve a special case of linear sixth order boundary value problems respectively. Lamnii et al. [10], kasi viswanadham and Showri raju [11] developed septic spline collocation and quintic B-spline collocation method are used to solve sixth order boundary value problems respectively. Loghmani and Ahmadinia [12] used sixth degree B-spline functions to construct an approximation solution for sixth order boundary value problems. Waleed [13] presented Adomian decomposition method with Green's function to solve a special case of sixth order boundary value problems. Kasi Viswanadham and Murali krishna [14] developed septic B-spline Collocation method to solve a special case of sixth order boundary value problems. Kasi Viswanadham and Sreenivasulu [15] developed quintic B-spline Galerkin method to solve a general sixth order boundary value problems. So far, sixth order boundary value problems have not been solved by using Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions. This motivated us to solve a sixth order boundary value problem by Pertrov-Galerkin method with quintic B-splines as basis functions and sextic B-splines as weight functions. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [16]. Finally, in the last section, the conclusions are presented.

JUSTIFICATION FOR USING PETROV-GALERKIN METHOD

In Finite Element Method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz method, Galerkin method, Least Squares method, Petrov-Galerkin method and Collocation method etc. In Petrov-Galerkin method, the residual of approximation is made orthogonal to the weight functions. When we use Petrov-Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [17, 18] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to the boundary conditions [19]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed and also the number of weight functions should match with the number of basis functions. Hence in this paper we employed the use of Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions to approximate the solution of sixth order boundary value problem.

DESCRIPTION OF THE METHOD

Definition of Quintic B-Splines and Septic B-Splines

The quintic B-splines and septic B-splines are defined in [20-22]. The existence of quintic spline interpolate s(x) to a function in a closed interval [c, d] for spaced knots (need not be evenly spaced) of a partition $c = x_0 < x_1 <\ldots< x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the quintic B-splines. Introduce ten additional knots x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} and x_{n+5} in such a way that

$x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}$.

Now the quintic B-splines $B_i(x)'s$ are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-3}^{i+3} \frac{(x_r - x)_+^5}{\pi'(x_r)}, & x \in [x_{i-3},\, x_{i+3}] \\ 0, & otherwise \end{cases}$$

where $$(x_r - x)_+^5 = \begin{cases} (x_r - x)^5, & if\ x_r \geq x \\ 0, & if\ x_r \leq x \end{cases}$$

and $$\pi(x) = \prod_{r=i-3}^{i+3} (x - x_r)$$

where $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x),\ldots,B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_5(\pi)$ of quintic polynomial splines. Schoenberg [22] has proved that quintic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0<x_1<\ldots<x_{n-1}<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}$.

In a similar analogue septic B-splines $R_i(x)$'s are defined by

$$R_i(x) = \begin{cases} \sum_{r=i-4}^{i+4} \frac{(x_r - x)_+^7}{\pi'(x_r)}, & x \in [x_{i-4},\, x_{i+4}] \\ 0, & otherwise \end{cases}$$

where $$(x_r - x)_+^7 = \begin{cases} (x_r - x)^7, & if\ x_r \geq x \\ 0, & if\ x_r \leq x \end{cases}$$

and $$\pi(x) = \prod_{r=i-4}^{i+4} (x - x_r)$$

where $\{R_{-3}(x), R_{-2}(x), R_{-1}(x), R_0(x), R_1(x),\ldots,R_{n-1}(x), R_n(x), R_{n+1}(x), R_{n+2}(x), R_{n+3}(x)\}$ forms a basis for the space $S_7(\pi)$ of septic polynomial splines with the introduction of four more additional knots x_{-7}, x_{-6}, x_{n+6}, x_{n+7} to the already existing knots x_{-5} to x_{n+5}. Schoenberg [22] has proved that septic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-7}<x_{-6}<x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0<x_1<\ldots<x_{n-1}<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}<x_{n+6}<x_{n+7}$.

To solve the boundary value problem (1) subject to boundary conditions (2) by the Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{3}$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are quintic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x),\ldots,B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and Neumann type of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of quintic B-splines, the Dirichlet and the Neumann boundary conditions of (2), we get the approximate solution at the boundary points as

$$A_0 = y(c) = y(x_0) = \sum_{j=-2}^{2} \alpha_j B_j(x_0) \tag{4}$$

$$C_0 = y(d) = y(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(x_n) \tag{5}$$

$$A_1 = y'(c) = y'(x_0) = \sum_{j=-2}^{2} \alpha_j B'_j(x_0) \tag{6}$$

$$C_1 = y'(d) = y'(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B'_j(x_n) \tag{7}$$

Eliminating α_{-2}, α_{-1}, α_{n+1} and α_{n+2} from the equations (3) to (7), we get

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j Q_j(x) \tag{8}$$

where

$$w(x) = w_1(x) + \frac{A_1 - w'_1(x_0)}{P'_{-1}(x_0)} P_{-1}(x) + \frac{C_1 - w'_1(x_n)}{P'_{n+1}(x_n)} P_{n+1}(x) \tag{9}$$

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x) \tag{10}$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P'_j(x_0)}{P'_{-1}(x_0)} P_{-1}(x), & j = 0, 1, 2 \\ P_j(x), & j = 3, 4, \ldots, n-3 \\ P_j(x) - \dfrac{P'_j(x_n)}{P'_{n+1}(x_n)} P_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{11}$$

$$P_j(x)=\begin{cases} B_j(x)-\dfrac{B_j(x_0)}{B_{-2}(x_0)}B_{-2}(x), & j=-1,0,1,2 \\ B_j(x), & j=3,4,\ldots,n-3 \\ B_j(x)-\dfrac{B_j(x_n)}{B_{n+2}(x_n)}B_{n+2}(x), & j=n-2,n-1,n,n+1 \end{cases} \tag{12}$$

The new set of basis functions in the approximation $y(x)$ is $\{Q_j(x), j=0,1,\ldots,n\}$. Here $w(x)$ takes care of given set of Dirichlet and Neumann type boundary conditions and $Q_j(x)$'s and its first order derivatives vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $y(x)$ defined in (8) is $n+1$, where as the number of weight functions is $n+7$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows:

Let us write the approximation for $v(x)$ as

$$v(x)=\sum_{j=-3}^{n+3}\beta_j R_j(x) \tag{13}$$

where $R_j(x)$'s are septic B-splines and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of the given boundary conditions (2). That means $v(x)$ defined in (13) satisfies the conditions

$$v(c)=0, v(d)=0, v'(c)=0, v'(d)=0, v''(c)=0, v''(d)=0 \tag{14}$$

Applying the boundary conditions (14) to (13), we get the approximate solution at the boundary points as

$$v(c)=v(x_0)=\sum_{j=-3}^{3}\beta_j R_j(x_0)=0 \tag{15}$$

$$v(d)=v(x_n)=\sum_{j=n-3}^{n+3}\beta_j R_j(x_n)=0 \tag{16}$$

$$v'(c)=v'(x_0)=\sum_{j=-3}^{3}\beta_j R'_j(x_0)=0 \tag{17}$$

$$v'(d)=v'(x_n)=\sum_{j=n-3}^{n+3}\beta_j R'_j(x_n)=0 \tag{18}$$

$$v''(c)=v''(x_0)=\sum_{j=-3}^{3}\beta_j R''_j(x_0)=0 \tag{19}$$

$$v''(d)=v''(x_n)=\sum_{j=n-3}^{n+3}\beta_j R''_j(x_n)=0 \tag{20}$$

Eliminating β_{-3}, β_{-2}, β_{-1}, β_{n+1}, β_{n+2} and β_{n+3} from the equations (13) and (15) to (20), we get the approximation for *v(x)* as

$$v(x)=\sum_{j=0}^{n}\beta_j V_j(x) \tag{21}$$

where

$$V_j(x)=\begin{cases} T_j(x)-\dfrac{T_j''(x_0)}{T_{-1}''(x_0)}T_{-1}(x), & j=0,1,2,3 \\ T_j(x), & j=4,5,\dots,n-4 \\ T_j(x)-\dfrac{T_j''(x_n)}{T_{n+1}''(x_n)}T_{n+1}(x), & j=n-3,n-2,n-1,n \end{cases} \tag{22}$$

$$T_j(x)=\begin{cases} S_j(x)-\dfrac{S_j'(x_0)}{S_{-2}'(x_0)}S_{-2}(x), & j=-1,0,1,2,3 \\ S_j(x), & j=4,5,\dots,n-4 \\ S_j(x)-\dfrac{S_j'(x_n)}{S_{n+2}'(x_n)}S_{n+2}(x), & j=n-3,n-2,n-1,n,n+1 \end{cases} \tag{23}$$

$$S_j(x)=\begin{cases} R_j(x)-\dfrac{R_j(x_0)}{R_{-3}(x_0)}R_{-3}(x), & j=-2,-1,0,1,2,3 \\ R_j(x), & j=4,5,\dots,n-4 \\ R_j(x)-\dfrac{R_j(x_n)}{R_{n+3}(x_n)}R_{n+3}(x), & j=n-3,n-2,n-1,n,n+1,n+2 \end{cases} \tag{24}$$

Now the new set of weight functions for the approximation *v(x)* is $\{V_j(x), j=0,1,\dots,n\}$. Here $V_j(x)$*'s* and its first and second order derivatives vanish on the boundary.

Applying the Petrov-Galerkin method to (1) with the new set of basis functions $\{Q_j(x), j=0, 1,\dots, n\}$ and the new set of weight functions $\{V_j(x), j=0,1,\dots,n\}$, we get

$$\begin{aligned}&\int_{x_0}^{x_n}[a_0(x)y^{(6)}(x)+a_1(x)y^{(5)}(x)+a_2(x)y^{(4)}(x)+a_3(x)y'''(x)+a_4(x)y''(x)+a_5(x)y'(x)\\ &+a_6(x)y(x)]V_i(x)\,dx=\int_{x_0}^{x_n}b(x)V_i(x)dx \quad \text{for } i=0,\ 1,\dots,n.\end{aligned} \tag{25}$$

Integrating by parts the first two terms on the left hand side of (25) and after applying the boundary conditions prescribed in (2), we get

$$\int_{x_0}^{x_n}a_0(x)V_i(x)y^{(6)}(x)dx=-\frac{d^3}{dx^3}\left[a_0(x)V_i(x)\right]_{x_n}C_2+\frac{d^3}{dx^3}\left[a_0(x)V_i(x)\right]_{x_0}A_2+\int_{x_0}^{x_n}\frac{d^4}{dx^4}\left[a_0(x)V_i(x)\right]y''(x)dx \tag{26}$$

$$\int_{x_0}^{x_n} a_1(x)V_i(x)y^{(5)}(x)dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3}\left[a_1(x)V_i(x)\right]y''(x)dx \tag{27}$$

Substituting (26) and (27) in (25) and using the approximation for $y(x)$ given in (8), and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{28}$$

where

$$\mathbf{A} = [a_{ij}];$$

$$a_{ij} = \int_{x_0}^{x_n} \{a_2(x)V_i(x)Q_j^{(4)}(x) + a_3(x)V_i(x)Q_j'''(x) + [\frac{d^4}{dx^4}[a_0(x)V_i(x)] - \frac{d^3}{dx^3}[a_1(x)V_i(x)]$$
$$+a_4(x)V_i(x)]Q_j''(x) + a_5(x)V_i(x)Q_j'(x) + a_6(x)V_i(x)Q_j(x)\}dx$$

for i= 0, 1, … , n; j=0, 1, … ,n. (29)

$$\mathbf{B} = [b_i];$$

$$b_i = \int_{x_0}^{x_n} \{b(x)V_i(x) - \{a_2(x)V_i(x)w^{(4)}(x) + a_3(x)V_i(x)w'''(x) + [\frac{d^4}{dx^4}[a_0(x)V_i(x)] - \frac{d^3}{dx^3}[a_1(x)V_i(x)]$$
$$+a_4(x)V_i(x)]w''(x) + a_5(x)V_i(x)w'(x) + a_6(x)V_i(x)w(x)\}\}dx + \frac{d^3}{dx^3}[a_0(x)V_i(x)]_{x_n} C_2 - \frac{d^3}{dx^3}[a_0(x)V_i(x)]_{x_0} A_2$$

for i=0, 1, ... ,n. (30)

and $\alpha = [\alpha_0\, \alpha_1 \ldots \alpha_n]^T$.

PROCEDURE TO FIND THE SOLUTION FOR NODAL PARAMETERS

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x)r_j(x)Z(x)dx$ and $r_j(x)$ are the quintic B-spline basis functions or their derivatives. $v_i(x)$ are the septic B-spline weight functions or their derivatives. It may be noted that I_m = *0* if $(x_{i-4}, x_{i+4}) \cap (x_{j-3}, x_{j+3}) \cap (x_m, x_{m+1}) = \varnothing$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiffness matrix $\mathbf{A}$ is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package. We have used the FORTRAN-90 program to solve the boundary value problems (1) - (2) by the proposed method.

NUMERICAL RESULTS

To demonstrate the applicability of the proposed method for solving the sixth order boundary value problems of the type (1) and (2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$y^{(6)} - 4y^{(4)} + 2y'' + xy = (5 + 2x - x^2)e^x, \quad 0 < x < 1 \tag{31}$$

subject to

$$y(0) = 1, \;\; y(1) = 0, \; y'(0) = 0, \; y'(1) = -e, \; y''(0) = -1, \; y''(1) = -2e.$$

The exact solution for the above problem is $y = (1 - x)e^x$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 1.382828x10^{-5}.

Table 1: Numerical Results for Example 1

x	Absolute Error by the Proposed Method
0.1	1.192093E-07
0.2	3.278255E-06
0.3	8.404255E-06
0.4	1.186132E-05
0.5	1.382828E-05
0.6	1.275539E-05
0.7	8.046627E-06
0.8	3.367662E-06
0.9	1.341105E-07

Example 2: Consider the linear boundary value problem

$$y^{(6)} + y^{(5)} + \sin x \quad y^{(4)} + xy = (2 + \sin x + x)e^x, \quad 0 < x < 1 \tag{32}$$

subject to

$$y(0) = 1, y(1) = e, \; y'(0) = 1, \; y'(1) = e, \; y''(0) = 1, \; y''(1) = e.$$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 1.716614x10^{-5}.

Table 2: Numerical Results for Example 2

x	Absolute Error by the Proposed Method
0.1	1.311302E-06
0.2	9.536743E-07
0.3	4.768372E-07
0.4	1.072884E-06
0.5	7.510185E-06

Table 2: Contd.,

0.6	1.549721E-05
0.7	1.716614E-05
0.8	1.406670E-05
0.9	9.775162E-06

Example 3: Consider the linear boundary value problem

$$y^{(6)} + y''' + y'' - y = (-15x^2 + 78x - 114)e^{-x}, \quad 0 < x < 1 \tag{33}$$

subject to

$$y(0) = 0, y(1) = \frac{1}{e}, y'(0) = 0, y'(1) = \frac{2}{e}, y''(0) = 0, y''(1) = \frac{1}{e}.$$

The exact solution for the above problem is $y = x^3 e^{-x}$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3. The maximum absolute error obtained by the proposed method is $2.533197\text{x}10^{-6}$.

Table 3: Numerical Results for Example 3

x	Absolute Error by the Proposed Method
0.1	3.114110E-08
0.2	1.401640E-07
0.3	1.527369E-07
0.4	3.352761E-08
0.5	9.238720E-07
0.6	2.115965E-06
0.7	2.533197E-06
0.8	2.190471E-06
0.9	1.639128E-06

Example 4: Consider the nonlinear boundary value problem

$$y^{(6)} + e^{-x} y^2 = e^{-x} + e^{-3x}, \quad 0 < x < 1 \tag{34}$$

subject to

$$y(0) = 1,\ y(1) = \frac{1}{e},\ y'(0) = -1,\ y'(1) = \frac{-1}{e},\ y''(0) = 1,\ y''(1) = \frac{1}{e}.$$

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (34) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [16] as

$$y^{(6)}_{(n+1)} + 2e^{-x} y_{(n)} y_{(n+1)} = e^{-x} y^2_{(n)} + e^{-x} + e^{-3x}, \quad n = 0, 1, 2, \ldots \tag{35}$$

subject to

$$y_{(n+1)}(0)=0,\ y_{(n+1)}(1)=\frac{1}{e},\ y'_{(n+1)}(0)=-1,\ y'_{(n+1)}(1)=-\frac{1}{e},\ y''_{(n+1)}(0)=1,\ y''_{(n+1)}(1)=\frac{1}{e}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (35). The obtained numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is $2.563000\text{x}10^{-6}$.

Table 4: Numerical Results for Example 4

x	Absolute Error by the Proposed Method
0.1	7.748604E-07
0.2	4.768372E-07
0.3	1.132488E-06
0.4	8.940697E-07
0.5	1.490116E-06
0.6	2.563000E-06
0.7	2.413988E-06
0.8	1.728535E-06
0.9	1.102686E-06

Example 5: Consider the nonlinear boundary value problem

$$y^{(6)}+y'y^{(5)}-\pi^3\sin(\pi x)y'''+yy''+\pi^2y^2=-\pi^6\cos(\pi x),\quad 0<x<1 \tag{36}$$

subject to

$$y(0)=1,\ y(1)=-1,\ y'(0)=0,\ y'(1)=0,\ y''(0)=-\pi^2,\ y''(1)=\pi^2.$$

The exact solution for the above problem is $y=\cos(\pi x)$.

The nonlinear boundary value problem (36) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [16] as

$$\begin{aligned} &y^{(6)}_{(n+1)}+y'_{(n)}y^{(5)}_{(n+1)}-\pi^3\sin(\pi x)y'''_{(n+1)}+y_{(n)}y''_{(n+1)}+y^{(5)}_{(n)}y'_{(n+1)} \\ &+(2\pi^2y_{(n)}+y''_{(n)})y_{(n+1)}=y_{(n)}y''_{(n)}+\pi^2y^2_{(n)}+y'_{(n)}y^{(5)}_{(n)}-\pi^6\cos(\pi x)\quad n=0,1,2,\ldots \end{aligned} \tag{37}$$

subject to

$$y_{(n+1)}(0)=1,\ y_{(n+1)}(1)=-1,\ y'_{(n+1)}(0)=0,\ y'_{(n+1)}(1)=0,\ y''_{(n+1)}(0)=-\pi^2,\ y''_{(n+1)}(1)=\pi^2.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (37). The obtained numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is $3.288842\text{x}10^{-5}$.

Table 5: Numerical Results for Example 5

x	Absolute Error by the Proposed Method
0.1	4.172325E-07
0.2	9.536743E-06
0.3	2.193451E-05
0.4	3.221631E-05

Table 5: Contd.,

0.5	3.288842E-05
0.6	2.413988E-05
0.7	1.323223E-05
0.8	3.933907E-06
0.9	7.748604E-07

Example 6: Consider the nonlinear boundary value problem

$$y^{(6)} - 20e^{-36y} = -40(1+x)^{-6}, \quad 0 < x < 1 \tag{38}$$

subject to

$$y(0)=0,\ y(1)=\frac{\ln 2}{6},\ y'(0)=\frac{1}{6},\ \ y'(1)=\frac{1}{12},\ y''(0)=-\frac{1}{6},\ y''(1)=-\frac{1}{24}.$$

The exact solution for the above problem is $y=\dfrac{\ln(1+x)}{6}$.

The nonlinear boundary value problem (38) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [16] as

$$y_{(n+1)}^{(6)} + 720e^{-36y_{(n)}}y_{(n+1)} = 720e^{-36y_{(n)}}y_{(n)} + 20e^{-36y_{(n)}} - 40(1+x)^{-6}, \quad n=0,1,2,... \tag{39}$$

subject to

$$y_{(n+1)}(0)=0, y_{(n+1)}(1)=\frac{\ln 2}{6}, y'_{(n+1)}(0)=\frac{1}{6}, y'_{(n+1)}(1)=\frac{1}{12}, y''_{(n+1)}(0)=-\frac{1}{6}, y''_{(n+1)}(1)=-\frac{1}{24}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (39). The obtained numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is $6.780028\text{x}10^{-07}$.

Table 6: Numerical Results for Example 6

x	Absolute Error by the Proposed Method Proposed Method
0.1	1.303852E-08
0.2	1.676381E-08
0.3	6.705523E-08
0.4	1.192093E-07
0.5	3.725290E-07
0.6	6.780028E-07
0.7	6.854534E-07
0.8	5.066395E-07
0.9	3.427267E-07

CONCLUSIONS

In this paper, we have employed a Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions to solve a general sixth order boundary value problems with special case of boundary conditions. The quintic B-spline basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and the Neumann boundary conditions are prescribed. The septic B-splines are redefined into a new set of weight functions which in number match the number of redefined set of basis functions. The proposed method has been tested on three linear and three nonlinear sixth order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The strength of the proposed method lies in its easy applicability, accurate and efficient to solve sixth order boundary value problems.

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